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Nonlinear Lagrangians and Einstein Spaces

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It is shown that, for a Riemannian space V_d of dimension d, solutions of the equation $\delta((-g)^{1/2}R^n)/\delta g^{ab} = 0$ for $n = (1/4)(d + 2)$ may be interpreted as $(d + 1)$-dimensional Einstein spaces.

1. INTRODUCTION

If R, R_{ab} are the curvature scalar and Ricci tensor, respectively, of a d-dimensional Riemannian space V_d $(d > 2)$ with metric g_{ab}, the functional derivative $\delta((-g)^{1/2}R^n)/\delta g^{ab}$ is given by[2]

$$R^{n-1}(nR_{ab} - \tfrac{1}{2}Rg_{ab}) - ng_{ab}\Box(R^{n-1}) + n(R^{n-1})_{;ab} = 0$$

In the following, $R \neq 0$ is assumed. We consider the equations $\delta((-g)^{1/2}R^n)/\delta g^{ab} = 0$ which may be rewritten in the form

$$R_{ab} + (n - 1)R^{-1}R_{;ab} + (n - 1)(n - 2)R^{-2}R_{;a}R_{;b} + \frac{1 - 2n}{2n(d - 1)}Rg_{ab} = 0 \tag{1a}$$

$$\Box R = \frac{d - 2n}{2n(n - 1)(1 - d)}R^2 + (2 - n)R^{-1}R_{;c}R_{;}{}^{c} \tag{1b}$$

For $n = 2$, in four-dimensional space, equations (1a, b) occasionally have been considered as candidates for replacing Einstein's field equations in gravitational theory.

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[2] Latin indices a, b, c run from 1 to d, while Greek indices range from 0 to d. If not otherwise indicated, indices are raised by g_{ab}. The semicolon denotes the covariant derivative with regard to g_{ab}; $\Box R := g^{ab}R_{;ab}$. A double stroke denotes the covariant derivative with regard to the metric $\bar{g}_{ab}$ of V_{d+1}.

The purpose of this note is twofold. First, a recent result of Buchdahl (1978) will be generalized. Then, it is shown that, for a certain value of n, solutions of the field equations (1a, b) may be interpreted as $(d + 1)$-dimensional static Einstein spaces.

2. LINEARIZATION OF FIELD EQUATIONS

In a $(d + 1)$-dimensional Riemannian space V_{d+1} the static metric is now considered

$$d\bar{s}^2 = R^{-2q}(dx^0)^2 + R^{2p}g_{ab}(x^c)\,dx^a\,dx^b \tag{2}$$

with p, q real and $R = R(x^c)$ the curvature scalar of V_d. Let $\bar{R} \coloneqq \bar{g}^{\alpha\beta}\bar{R}_{\alpha\beta}$ and $\bar{R}_{\alpha\beta}$ be the curvature scalar and Ricci tensor of V_{d+1} with the metric (2). From the general formulas given in Buchdal (1954)

$$\bar{R}_{00} = R^{-2(p+q)}\{-qR^{-1}\Box R + R^{-2}R_{;c}R_{;}{}^{c}q[1 + q - p(d - 2)]\} \tag{3a}$$

$$\bar{R}_{0a} = 0 \tag{3b}$$

$$\begin{aligned}\bar{R}_{ab} = {} & R_{ab} + [(d - 2)p - q]R^{-1}R_{;ab} \\ & + [q(q + 1) + 2pq - p(p + 1)(d - 2)]R^{-2}R_{;a}R_{;b} \\ & + g_{ab}\{pR^{-1}\Box R + p[p(d - 2) - q - 1]R^{-2}{}_{;c}R_{;}{}^{c}\}\end{aligned} \tag{3c}$$

$$\begin{aligned}\bar{R} = {} & R^{-2p}\{R + 2R^{-1}\Box R(dp - p - q) + R^{-2}R_{;c}R_{;}{}^{c} \\ & \times [-dp + 2q(q + 1) + (d - 2)(dp^2 - 2pq - p - p^2)]\}\end{aligned} \tag{3d}$$

By use of equations (3a–d) one can show that equations (1a, b) and

$$\bar{R}_{ab} + k\bar{g}_{ab}\bar{R} = 0 \tag{4}$$

are in accord if k, n, and p are chosen properly. In fact, (4) is equivalent to

$$\begin{aligned}& R_{ab} + [(d - 2)p - q]R^{-1}R_{;ab} + [q(q + 1) - p(p + 1)(d - 2) + 2pq]R^{-2}R_{;a}R_{;b} \\ & \quad + N^{-1}[p + k(pd - q)]g_{ab}\{-R + p(d - 1)[p(d - 2) - 2q]R^{-2}R_{;c}R_{;}{}^{c}\} = 0\end{aligned} \tag{5}$$

where

$$N \coloneqq q + 2(1 + kd)(pd - p - q)$$

Comparison of equations (1a) and (5) leads to

$$n = q + 1, \qquad p = 2q(d - 2)^{-1} \tag{6a}$$

$$k = \tfrac{1}{2}[d - 2 - 2dq][d^2q - (1 + q)(d - 2)]^{-1} \tag{6b}$$

The contracted Bianchi identities, in V_{d+1}, $\bar{R}^{\alpha}_{\beta\|\alpha} = 0$ after integration lead to the following expression of $\bar{R}$ as a function of R:

$$\bar{R} = q(d - 2)^2M^{-1}R^S \tag{7}$$

where

$$S := [d - 2 - 4q](d - 2)^{-1}$$
$$M := d(d - 2)(2k + 1) + q(d - 2)^2 - 2qd(kd + 2k + 2)$$

For $d = 4$, $n = 2$, i.e., a quadratic Lagrangian in a four-dimensional space, the result of Buchdahl (1978) is recovered.

3. (d + 1)-DIMENSIONAL EINSTEIN SPACES

By a straightforward calculation using equations (3a, d) one concludes that

$$\bar{R}_{00} + k\bar{g}_{00}\bar{R} = 0 \tag{8}$$

is consistent with (the trace of) equation (4) if and only if

$$k = -(1 + d)^{-1} \tag{9}$$

Equations (6b) and (9) then lead to

$$q = \tfrac{1}{4}(d - 2) \tag{10a}$$

If this value of q is substituted in (6a), n and p take the values

$$n = \tfrac{1}{4}(d + 2), \qquad p = \tfrac{1}{2} \tag{10b}$$

In this case, the metric $\bar{g}_{\alpha\beta}$ becomes

$$d\bar{s}^2 = R^{1-d/2}(dx^0)^2 + Rg_{ab}(x^c)\, dx^a\, dx^b \tag{2'}$$

and V_{d+1} is an Einstein space. If (9) and (10a) hold, from (7) $\bar{R}$ = const. follows as required. It is not difficult to see that g_{ab} cannot be the metric of an Einstein space V_d, too.

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